MATH2050C Assignment 5

Section 3.3 no. 3, 5, 7, 10.

Section 3.3

(5) $y_1 = \sqrt{p}, p > 0$, and $y_{n+1} = \sqrt{p + y_n}$. Use induction it is straightforward to see $\{y_n\}$ is increasing. To show boundedness we follow the hint and use induction to show $y_n \le 1 + 2\sqrt{p}$. Assuming $y_n \le 1 + 2\sqrt{p}$, we have

$$y_{n+1}^2 = p + y_n \le p + 1 + 2\sqrt{p} = (1 + \sqrt{p})^2,$$

hence

$$y_{n+1} \le 1 + \sqrt{p} < 1 + 2\sqrt{p} \ .$$

(7) It is clear that $x_{n+1} = x_n + 1/x_n, x_1 > 0$, is increasing. Were it bounded from above, its limit exists by Monotone Convergence Theorem. Letting the limit be $b > 0$, then passing limit in the defining relation of the sequence we get $b = b + 1/b$, which is ridiculous. We conclude that $\{x_n\}$ is divergent to infinity.

(10). We claim the sequence $\{y_n\}$ given by

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n},$$

is increasing and bounded. First, we have

$$y_n < \frac{1}{n} + \frac{1}{n} + \cdots + \frac{n}{n} = \frac{n}{n} = 1 \ , \quad \forall n \ge 1,$$

hence $\{y_n\}$ is bounded from above. Next,

$$y_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} \ .$$

We have

$$y_{n+1} - y_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} > 0 \ , \quad \forall n \ge 1 \ ,$$

hence it is increasing. By Monotone Convergence Theorem $\{y_n\}$ is convergent.

Note. One can show that the limit is $\log 2$.

Supplementary Problems

1. Show that the sequence $\{b_n\}, b_n = \sum_{k=1}^{n} \frac{1}{k^a}$ is convergent iff and only if $a > 1$. Hint: Study b_{2^n} as in Example 3.3.3b in textbook.

Solution For $a > 1$,

$$\begin{aligned}
b_{2^n} &= 1 + \frac{1}{2^a} + \Big(\frac{1}{3^a} + \frac{1}{4^a}\Big) + \cdots + \Big(\frac{1}{(2^{n-1}+1)^a} + \cdots + \frac{1}{2^{an}}\Big) \\
&< 1 + \frac{1}{2^a} + \frac{2}{3^a} + \cdots + \frac{2^n}{(2^{n-1}+1)^a} \\
&< 1 + \frac{1}{2^a} + \frac{2}{2^a} + \cdots + \frac{2^n}{2^{a(n-1)}} \\
&< 1 + \frac{1}{2^a} + \frac{2}{2^a}\sum_{k=0}^{\infty} \frac{1}{2^{(a-1)k}} \\
&= 1 + \frac{1}{2^a} + \frac{2}{2^a}\frac{1}{1-2^{1-a}} \ ,
\end{aligned}$$

which shows that b_{2^n} is bounded above. Since b_n is increasing, the entire sequence is bounded above, hence it is convergent by the Monotone Convergence Theorem. The divergence case is immediate since $1/n^a > 1/n$ when $a < 1$ and $\sum_n 1/n$ is divergent.

2. Show that (a) $x_n = (1 + 1/n)^n$ is strictly increasing and $y_n = (1 + 1/n)^{n+1}$ is strictly decreasing. Hint: Try the Bernoulli inequality.

Solution We have

$$\begin{aligned}
\frac{x_{n+1}}{x_n} &= \Big(1 + \frac{1}{n+1}\Big)\Big(\frac{1 + 1/(n+1)}{1 + 1/n}\Big)^n \\
&= \Big(1 + \frac{1}{n+1}\Big)\Big(1 - \frac{1}{(n+1)^2}\Big)^n \\
&> \Big(1 + \frac{1}{n+1}\Big)\Big(1 - \frac{n}{(n+1)^2}\Big) \\
&= 1 + \frac{1}{(n+1)^3} \\
&> 1 \ ,
\end{aligned}$$

so $\{x_n\}$ is strictly increasing. Next,

$$\begin{aligned}
\frac{y_n}{y_{n-1}} &= \Big(1 + \frac{1}{n}\Big)\Big(\frac{1}{1 + 1/(n^2-1)}\Big)^n \\
&< \Big(1 + \frac{1}{n}\Big)\frac{1}{1 + n/(n^2-1)} \\
&= \frac{n^3 + n^2 - n - 1}{n^3 + n^2 - n} \\
&< 1 \ ,
\end{aligned}$$

so $\{y_n\}$ is strictly decreasing.

Remark It is clear that $x_n < y_n$. In particular, x_n is bounded by $y_1 = 4$. Then we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = e$. This provides an alternative proof of the existence of $\lim_n (1 + 1/n)^n$.

3. Show the limit of $(1 - 1/n)^n$ exists. Hint: Use Problem 3 in Ex 4.

Solution We have

$$\lim_{n\to\infty}\Big(1 - \frac{1}{n}\Big)^n = \frac{\lim_{n\to\infty}\big(1 - 1/n^2\big)^n}{\lim_{n\to\infty}\big(1 + 1/n\big)^n} = \frac{1}{e} \ .$$

4. Prove that e is irrational. Hint: Use the inequality $0 < e-(1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{k!}) < \frac{1}{k\times k!}$.

   Solution Suppose on the contrary that $e = p/q$, a rational number. Then taking $k = q \geq 2$ in the inequality to get

$$0 < p(q-1)! - q!(1+1+1/2!+\cdots+1/k!) < 1/q \leq 1/2.$$

   Noting that $q!(1+1+1/2!+\cdots+1/k!)$ is a natural number, it is impossible to have two distinct natural numbers whose difference is less than $1/2$. Contradiction holds.